15. Exercises

Exercise 15.1. Prove Proposition 3.2.

Exercise 15.2. Since we have not said what formal system of deduction we are using, the proof above of Proposition 3.3 tacitly appeals several times to soundness and completeness. Re-write the argument to make those appeals explicit.

Exercise 15.3. Show that '$\forall x(0 + x = x)$' is not a theorem of $\boldsymbol{Q}$. (Hint: The domain will need to be expanded, so that it contains not just the natural numbers but also some 'rogue' objects for which addition behaves strangely.) The model you give is likely to falsify other arithmetical truths, or easily to be adapatable so that it does so. Play around with this a bit, and prove that $\boldsymbol{Q}$ doesn't prove some other things.

Exercise 15.4. It follows from 5.1 that $\boldsymbol{Q}$ proves every true equality of the form '$t = u$', and every true inequality of the forms '$t \neq u$' and $t < u$, where t and u are arbitrary terms of the language of arithmetic. Prove this fact (by induction on the complexity of these terms).

Exercise 15.5. Prove proposition 5.4. (Hint: You will need to use an instance of the induction axiom.)

Exercise 15.6. Show that PA proves the commutativity of addition, i.e., that $\forall x \forall y(x + y = y + x)$. The easiest proof uses induction on y, so you want to show is that PA proves $\forall x(x+0 = 0+x)$ and $\forall x(x+n = n+x) \rightarrow \forall x(x + Sn = Sn + x)$.

Exercise 15.7. Show that condition (ii) in Definition 6.3 actually follows from condition (i).

Exercise 15.8. Prove Proposition 6.7.

Exercise 15.9. Prove Theorem 6.8 using Theorem 4.6.

Exercise 15.10. Prove Corollary 8.2.

Exercise 15.11. Let Σ be an arbitrary theory. Prove the following facts about its closure:

(i) $Cl(\Sigma)$ is closed.

(ii) $Cl(\Sigma) = Cl(Cl(\Sigma))$.

(iii) If $\Theta \supseteq \Sigma$ is closed, then $\Theta \supseteq Cl(\Sigma)$. (Hint: Show that if $\Sigma \subseteq \Delta$, then $Cl(\Sigma) \subseteq Cl(\Delta)$, and then apply (ii).