Exercises

Exercise 2.1. Show that the functions

$$\beta(x, y) = \begin{cases} 1 & \text{, if } x > 0 \wedge y > 0 \\ 0 & \text{, otherwise} \end{cases}$$

and

$$\gamma(x, y) = \begin{cases} 0 & \text{, if } x = y \\ 1 & \text{, if } x \neq y \end{cases}$$

are (primitive) recursive. Explain how we can then use γ to prove that if $\phi(\vec{x})$ and $\psi(\vec{y})$ are recursive functions, then the relation $\phi(\vec{x}) = \psi(\vec{y})$ is recursive.

Exercise 2.2. Gödel uses the fact that the relation $x < y$ is recursive, but does not prove this fact (or even mention that it needs proving). Prove it. (There's an easy proof that uses Gödel's Theorem IV and the fact that addition is recursive. There's a better proof that does not use these resources.)

Exercise 2.3. Gödel does not include '$=$' in the language of his theory *P*, because it can be defined in simple type theory as:

$$u_n = v_n \equiv \forall t_{n+1}(t_{n+1}(u_n) \rightarrow t_{n+1}(v_n))$$

The equivalent definition for objectual identity in second-order logic is:

$$a = b \equiv \forall F(Fa \rightarrow Fb)$$

You may do the various parts of this exercise using either definition.

(i) Show that this definition is equivalent to the apparently stronger definition in which '$\rightarrow$' has been replaced by '$\equiv$'.

(ii) Show that the other basic properties of identity

$$a = a$$
$$a = b \rightarrow b = a$$
$$a = b \wedge b = c \rightarrow a = c$$

can all be derived from the definition.

Exercise 2.4. In the proof of Theorem IV, Gödel defines a function

$$\chi(z, \vec{y}) = \begin{cases} 0 & \text{, if } z < n \\ n & \text{, if } n \leq z \end{cases}$$

where n is the least number for which $R(n, \vec{y})$ holds, if any, and χ is constantly 0 if there is no such n. The definition is as follows:

$$\chi(0, \vec{y}) = 0$$
$$\chi(n+1, \vec{y}) = a(n) \cdot (n+1) + \alpha(a(n)) \cdot \chi(n, \vec{y})$$

where:

$$a(n) = \alpha(\alpha(\rho(0, \vec{y})) \cdot \alpha(\rho(n+1, \vec{y})) \cdot \alpha(\chi(n, \vec{y}))$$

$$\alpha(x) = \begin{cases} 0 & \text{, if } x \neq 0 \\ 1 & \text{, if } x = 0 \end{cases}$$

and $\rho(z, \vec{y})$ is the characteristic function of the relation $R(z, \vec{y})$. Explain why the definition of χ works.

Exercise 2.5. Gödel's coding of finite sequences works only for positive integers. Let's explore this situation.

(i) If we try to code the sequence $<2, 0, 2>$ Gödel's way, we get $2^2 \cdot 3^0 \cdot 5^2 = 100$. But his definition of $n\, Gl\, s$ tells us that $1\, Gl\, 100 = 2$, $2\, Gl\, 100 = 2$, and $3\, Gl\, 100 = 0$, which is wrong. Explain why.

(ii) We can fix this problem by slightly changing the coding, so that $<3, 5, 7>$ is coded as $2^4 \cdot 3^6 \cdot 5^8$, and $<2, 0, 2>$ can then be coded as $2^3 \cdot 3^1 \cdot 5^3$. How would we have to change Gödel's definition (6) to make this work?

(iii) On neither treatment does every number code a sequence. Which numbers do code sequences? Show that the property of *being a number that codes a sequence* is (primitive) recursive by defining it in the same sort of way Gödel defines his 45 notions. You may do so for either of the two codings.

Exercise 2.6. Gödel's definition of concatenation is a bit magical. Show that

$$\begin{aligned} \text{Cat}(s, t, 0) &= s \\ \text{Cat}(s, t, n+1) &= \text{Cat}(s, t, n) \cdot Pr(l(\text{Cat}(s, t, n)) + 1)^{(n+1)\, Gl\, t} \\ s * t &= \text{Cat}(s, t, l(t)) \end{aligned}$$

also works, and explain why maybe it isn't so magical.

Exercise 2.7. Show that the successor function $Sx = x + 1$ and the various constant functions $c_n(x) = n$ are arithmetically definable, in Gödel's sense: They are definable in (first-order) logic in terms of the two operations of addition and multiplication.

Exercise 2.8. Show that the function x mod y is arithmetically definable, i.e., that there is an arithmetical formula $\text{Mod}(x, y, z)$ such that $\text{Mod}(\overline{s}, \overline{d}, \overline{z})$ is true iff $z = x$ mod y.

Exercise 2.9 (Optional)**.** Say that a set A is 'additive' if it satisfies the following two conditions:

$$<x, 0, x> \in A$$
$$<x, y, z> \in A \rightarrow <x, Sy, Sz> \in A$$

Let

$$\text{sum} \equiv \{<x, y, z> \in \bigcap \{A : A \text{ is an additive set}\}$$

Show that sum defines addition, i.e., that $< x, y, z > \in$ sum iff $x + y = z$. (Hint: The two conditions on additive sets correspond precisely to the two recursion equations for addition.)