EXERCISES ON TARSKI, MOSTOWSKI, AND ROBINSON, AND BOOLOS

Exercise 3.1. Prove, as carefully as possible, and by appeal to Tarski's definition of interpretability, that: (i) Every theory is interpretable in itself; and (ii) If S is interpretable in T and T is interpretable in U, then S is interpretable in U. Prove that the same is true of relative interpretability.

Exercise 3.2. A theory T is said to be *locally* (relatively) interpretable in a theory B if every finite sub-theory of T is (relatively) interpretable in B. (Obviously, this notion is useful only if T is infinite.) Show that, if T is locally (relatively) interpretable in B so, then T is consistent if B is.

Exercise 3.3. Tarski, Mostowski, and Robinson claim on p. 56 that, if there is a formula $F(x_1, \ldots, x_n, y)$ that meets conditions (i) and (ii) in their definition of representability, then the formula

$$F(x_1, \ldots, x_n, y) \wedge \forall z[F(x_1, \ldots, x_n, z) \rightarrow y \leq z]$$

will meet all three of the conditions. Show that this is true. Does the argument go through in ***R***, or do we have to assume something stronger about the theory in which we are working?

Exercise 3.4. Show that none of the axioms of ***Q*** are provable in ***R*** and, indeed, that even the *disjunction* of the axioms of ***Q*** is not provable in ***R***.

Exercise 3.5.

(i) Tarski, Mostowski, and Robinson claim, but do not prove, that ***R*** is not finitely axiomatizable. Show that it is not by proving the following even stronger claim: Let ***S*** be any sub-theory of ***R*** that contains as axioms all instances of (R1)–(R4), but only finitely many instances of (R5). Then there are instances of (R5) that are not provable in ***S***.

(ii) For which of (R1)–(R4) can one prove the analogue of (i)?

Exercise 3.6. (Optional) Show that ***R*** is 'locally finitely satisfiable', meaning: For every finite subset of the axioms of ***R***, there is a *finite* model that makes them all true.

Exercise 3.7. In the proof of "Provable $\Sigma_1$completeness", on pp. 46–8, Boolos leaves many cases unproved. In particular, in the basis of the induction, on p. 46, he proves only the case for addition. Prove the other four. In the inductive part, on p. 47, Boolos also does not do the case of disjunction. Do that one.