Exercises on Feferman

Exercise 4.1. Give a careful proof of Feferman's Theorem 4.5. (Just the first two parts.) You may assume that $Sq(x)$ bi-numerates the set of Gödel numbers of sequences, and similarly for other notions that occur in the definition of Prf_α.

Exercise 4.2. Prove parts (i) and (iii) of Feferman's Theorem 4.7, and part (i) of Theorem 4.8.

Exercise 4.3. Prove the first statement in Feferman's Theorem 4.15. (The part after "hence" then follows from 4.8(i).) For an optional bonus: Explain the remarks made just before 4.15, and show that they are correct.

Exercise 4.4. Give a detailed proof of Feferman's Theorem 5.3.

Exercise 4.5. Feferman claims on p. 32 that no (consistent) reflexive theory that is subject to the second incompleteness theorem is finitely axiomatizable. This isn't as obvious as it seems. What follows immediately is just that, if $\mathcal{T}$ is reflexive, then it cannot be axiomatized by any finite subset of $\mathcal{T}$ itself. (I.e., for no finite $\mathcal{U} \subseteq \mathcal{T}$ do we have $\mathcal{U} \vdash A$ iff $\mathcal{T} \vdash A$.) But mightn't there be some *other* finite set of formulas that axiomatizes $\mathcal{T}$? Well, no. But why not? Answer the question by proving the following:

> Let $\mathcal{T}$ be an infinite set for sentences, and suppose that there is a finite set $\mathcal{T}'$ such that $\mathcal{T} \vdash A$ iff $\mathcal{T}' \vdash A$. Then for some finite $\mathcal{U} \subseteq \mathcal{T}$, $\mathcal{T} \vdash A$ iff $\mathcal{U} \vdash A$.