Philosophy 1880
Series 7
Soundness and Completeness

Say some stuff about the type of sequent calculus used in the book, and then introduce this one, which is a variant of one Burgess mentions. We'll do everything for it.

1 Rules

$A \Rightarrow A$

If $\Gamma \Rightarrow A$ and $\Delta \supseteq \Gamma$, then $\Delta \Rightarrow A$

(¬+)

$\Gamma, A \Rightarrow B$
$\Delta, A \Rightarrow \neg B$
$\therefore \Gamma, \Delta \Rightarrow \neg A$

(¬-)

$\Gamma \Rightarrow \neg\neg A$
$\therefore \Gamma \Rightarrow A$

(∨+)

$\Gamma \Rightarrow A$
$\therefore \Gamma \Rightarrow A \vee B$

(∨+)

$\Gamma \Rightarrow B$
$\therefore \Gamma \Rightarrow A \vee B$

(∨-)

$\Gamma, A \Rightarrow C$
$\Delta, B \Rightarrow C$
$\Theta \Rightarrow A \vee B$
$\therefore \Gamma, \Delta, \Theta \Rightarrow C$

(∧+)

$\Gamma \Rightarrow A$
$\Delta \Rightarrow B$
$\Gamma, \Delta \Rightarrow A \wedge B$

(∧-)

$\Gamma \Rightarrow A \wedge B$
$\therefore \Gamma \Rightarrow A$

(∧-)

$\Gamma \Rightarrow A \wedge B$
$\therefore \Gamma \Rightarrow B$

$$\begin{array}{ll} \Gamma, A \Rightarrow B & (\to+) \\ \therefore \Gamma \Rightarrow A \to B & \end{array}$$

$$\begin{array}{ll} \Gamma \Rightarrow A \to B & (\to\text{-}) \\ \Delta \Rightarrow A & \\ \therefore \Gamma, \Delta \Rightarrow B & \end{array}$$

$$\begin{array}{ll} \emptyset \Rightarrow t = t & (=+) \end{array}$$

$$\begin{array}{ll} \Gamma \Rightarrow A(s) & (=\text{-}) \\ \Delta \Rightarrow s = t & \\ \therefore \Gamma, \Delta \Rightarrow A(t) & \end{array}$$

$$\begin{array}{ll} \Gamma \Rightarrow A(t) & (\exists+) \\ \Gamma \Rightarrow \exists x A(x) & \end{array}$$

$$\begin{array}{ll} \Gamma, A(y) \Rightarrow B & (\exists\text{-}) \\ \Delta \Rightarrow \exists x A(x) & \\ \therefore \Gamma, \Delta \Rightarrow B & \end{array}$$

$$\begin{array}{ll} \Gamma \Rightarrow A(y) & (\forall+) \\ \therefore \Gamma \Rightarrow \forall x A(x) & \end{array}$$

$$\begin{array}{ll} \Gamma \Rightarrow \forall x A(x) & (\forall\text{-}) \\ \Gamma \Rightarrow A(t) & \end{array}$$

Note: In the identity and quantifier rules, t and s can be any term, subject to restrictions on capturing; y can be any constant, subject to the condition, in $\exists-$, that it not be free in Γ or in B, and in $\forall+$, that it not be free in Γ.

2 Soundness

Theorem (Soundness)**.** *If* $\Gamma \Rightarrow A$*, then* $\Gamma \vDash A$.

Proof. The proof is by induction on the length of proofs. Say that a sequent $\Gamma \Rightarrow A$ is *secure* $\Gamma \vDash A$. We shall show that all correct proofs have secure sequents as each of their steps; so, in particular, the last sequent will be secure, and so the conclusion of the proof, $\Gamma \Rightarrow A$, will be secure, whence $\Gamma \vDash A$, as wanted.

We show first that all one-line proofs satisfy this condition. A one-line proof can only consist of a basic sequent, $A \Rightarrow A$. But of course $A \vDash A$.

Suppose, then, that all n line proofs satisfy the condition. We want to show that all $n+1$ line proofs also satisfy it. We have to consider several different cases, one for each of the rules of our system. We shall not consider all the rules, but shall limit our attention the basic rules and those for $\neg$, $\vee$, and $\exists$. The other cases are similar.

If line $n+1$ is a basic sequent, then it is secure for the reason already given.

If line $n+1\ldots$

- follows by thinning, then it is $\Gamma \Rightarrow A$ and there is some earlier line that is $\Delta \Rightarrow A$, where $\Gamma \supseteq \Delta$. By the i.h., $\Delta \Rightarrow A$ is secure, so $\Delta \vDash A$. But then $\Gamma \vDash A$...

- follows by $\vee+$, then it is $\Gamma \Rightarrow A \vee B$ and some earlier line is either $\Gamma \Rightarrow A$ or $\Gamma \Rightarrow B$. Suppose the former. Then it is secure, so $\Gamma \vDash A$. But $A \vDash A \vee B$, whence $\Gamma \vDash A \vee B$...

- follows by $\vee-$, then it is $\Gamma, \Delta, \Theta \Rightarrow C$, and we have as earlier lines

$$\begin{aligned} \Gamma, A &\Rightarrow C \\ \Delta, B &\Rightarrow C \\ \Theta &\Rightarrow A \vee B \end{aligned}$$

  These are all secure. So suppose $\mathscr{M}$ makes (all the sentences) in Γ, Δ, and Θ true. Since Θ implies $A \vee B$, $\mathscr{M}$ makes $A \vee B$ true; so it must either make A true or else make B true. In the former case, all the sentences in Γ are true, and so is A, so C must be true, and something is similar in the latter case. So C just be true in any interpretation that makes Γ, Δ, and Θ true. That is: $\Gamma, \Delta, \Theta \vDash C$...

- follows by $\neg+$, then it is $\Gamma, \Delta \Rightarrow \neg A$ and we have, on some earlier lines:

$$\begin{aligned} \Gamma, A &\Rightarrow B \\ \Delta, A &\Rightarrow \neg B \end{aligned}$$

  and these are secure. Let $\mathscr{M}$ be an interpretation that makes both Γ and Δ true. Suppose it also makes A true. Since it makes Γ true, it must make B true, since $\Gamma, A \vDash B$. Since it makes Δ true, it must make $\neg B$ true, since $\Delta, A \vDash \neg B$. But that is impossible, any interpretation that makes both Γ and Δ makes A false. Thus, $\Gamma, \Delta \vDash \neg A$...

- follows by $\neg-$, then it is $\Gamma \Rightarrow A$, and we have as some earlier line $\Gamma \Rightarrow \neg\neg A$, and this is secure, i.e., $\Gamma \vDash \neg\neg A$. Since $\neg\neg A \vDash A$, we have $\Gamma \vDash A$...

- follows by $\exists+$, then it is $\Gamma \Rightarrow \exists x A(x)$ and some earlier line is $\Gamma \Rightarrow A(t)$, which is secure. So $\Gamma \vDash A(t)$. But $A(t) \vDash \exists x A(x)$. Hence, $\Gamma \vDash \exists x A(x)$...

- follows by $\exists-$, then it is $\Gamma, \Delta \Rightarrow B$, and we have as earlier lines

$$\begin{aligned} \Gamma, A(y) &\Rightarrow B \\ \Delta &\Rightarrow \exists x A(x) \end{aligned}$$

  and y is not free in Γ or in B. These are secure.
  Let $\mathscr{M}$ be an interpretation that makes Γ and Δ true. Since it makes Δ true, it makes $\exists x A(x)$ true. So if z is a new constant, which we may assume is not present in Γ, or in Δ, or in B, then there is an interpretation $\mathscr{N}$ that differs

from $\mathscr{M}$, if at all, only in what it assigns as value to z such that $\mathscr{N} \vDash A(z)$. Since z is not contained in Γ, $\mathscr{N}$ still makes Γ true. Now let $\mathscr{K}$ be the interpretation that is just like $\mathscr{N}$ except that it assigns also to y whatever $\mathscr{N}$ assigns to z. Since y is not contained in Γ, $\mathscr{K}$ still makes Γ true. Moreover, $\mathscr{K}$ makes $A(y)$ true, since $\mathscr{K}$ still makes $A(z)$ true and $A(y)$ contains y exactly where $A(z)$ contains z and these constants have the same value in $\mathscr{K}$. Thus, $\mathscr{K}$ makes B true, since $\Gamma, A(y) \vDash B$. But now $\mathscr{K}$ differs from $\mathscr{M}$ only in what it assigns to y and z, and neither of these is present in B. So $\mathscr{M}$ also makes B true, in which case $\Gamma, \Delta \vDash B \ldots$

- follows by $=+$, then it is $\emptyset \Rightarrow t = t$. But, since $t = t$ is valid, certainly $\emptyset \vDash t = t \ldots$

- follows by $=-$, then it is $B(t)$, and we have as earlier lines

$$\Gamma \Rightarrow B(s)$$
$$\Delta \Rightarrow s = t$$

  and these are secure. Let $\mathscr{M}$ be an interpretation that makes both Γ and Δ true. So it makes both $B(s)$ and $s = t$ true, which is to say that s and t have the same value in $\mathscr{M}$. But then $\mathscr{M}$ must also make $B(t)$ true, by the Extensionality Lemma (Proposition 10.2, in the book). That is: $\Gamma, \Delta \vDash B(t) \ldots$

so this line, too, is secure. Since we already knew all the previous lines were secure, all lines of this $n+1$ line proof are secure, as promised. □

3 Completeness

Theorem (Completeness)**.** *If* $\Gamma \vDash A$*, then* $\Gamma \Rightarrow A$*.*

3.1 Reducing the Problem

Definition. A set Γ is (deductively) *consistent* iff $\Gamma \not\Rightarrow A \wedge \neg A$, for any A.

Let $\bot$ be an abbreviation for some fixed contradictory sentence, e.g., $\forall x(x = x) \wedge \neg\forall x(x = x)$. We shall see shortly that we have as 'derived rules' of our system:

$$\Gamma \Rightarrow A \wedge \neg A \qquad (\bot+)$$
$$\therefore \Gamma \Rightarrow \bot$$

$$\Gamma \Rightarrow \bot \qquad (\bot-)$$
$$\therefore \Gamma \Rightarrow A$$

Lemma 1. $\Gamma \Rightarrow A$ *iff* $\Gamma, \neg A \Rightarrow \bot$*, and* $\Gamma \Rightarrow \neg A$ *iff* $\Gamma, A \Rightarrow \bot$

Proof. We prove the first half:

$$\begin{aligned} \Gamma &\Rightarrow A & \\ \neg A &\Rightarrow \neg A & \\ \Gamma, \neg A &\Rightarrow A \wedge \neg A & (\wedge+) \\ \Gamma, \neg A &\Rightarrow \bot & (\bot+) \end{aligned}$$

$$\begin{aligned} \Gamma, \neg A &\Rightarrow \bot & \\ \Gamma, \neg A &\Rightarrow \forall x(x = x) \wedge \neg\forall x(x = x) & (\text{def } \bot) \\ \Gamma, \neg A &\Rightarrow \forall x(x = x) & (\wedge\text{-}) \\ \Gamma, \neg A &\Rightarrow \neg\forall x(x = x) & (\wedge\text{-}) \\ \Gamma &\Rightarrow \neg\neg A & (\neg+) \\ \Gamma &\Rightarrow A & (\neg\text{-}) \end{aligned}$$

The second half is similar. □

So Σ is consistent iff $\Sigma \not\Rightarrow \bot$.

Let Σ be the set of consistent sets of sentences. If we can prove that Σ has the satisfaction properties, then it will follow from the term models lemma that every set of sentences in Σ has a model. So we can then argue as follows:

$$\begin{aligned} \Gamma \Rightarrow A &\Rightarrow \Gamma \cup \{\neg A\} \text{ has no model} \\ &\Rightarrow \Gamma \cup \{\neg A\} \text{ is inconsistent} \\ &\Rightarrow \Gamma \Rightarrow A \end{aligned}$$

thus establishing completeness.

3.2 Some Derived Rules

It will be helpful in showing that the set of consistent sets of sentences has the satisfaction properties to establish some 'derived rules' of the system. A 'derived rule' is not a real rule, but we can act as if it were, because we can always get the effect of the derived rule by a sequence of other steps.

Lemma 2. *If* $A \in \Gamma$*, then* $\Gamma \Rightarrow A$.

Proof.

$$\begin{aligned} A &\Rightarrow A & \\ \Gamma &\Rightarrow A & (\text{thinning}) \end{aligned}$$

Thinning can be applied here since, if $A \in \Gamma$, then $\{A\} \subseteq \Gamma$. □

Lemma 3 (Cut)**.** *The following are 'derived rules' of the system:*

$$\begin{array}{rr} \Gamma \Rightarrow A & \Gamma, A, B \Rightarrow C \\ \Delta, A \Rightarrow B & \Delta, A \Rightarrow B \\ \therefore \Gamma, \Delta \Rightarrow B & \therefore \Gamma, \Delta, A \Rightarrow C \end{array}$$

Proof. For the first:

$$\begin{array}{rlr} \Gamma \Rightarrow A & & \\ \Delta, A \Rightarrow B & & \\ \Delta \Rightarrow A \to B & & (\to+) \\ \Gamma, \Delta \Rightarrow B & & (\to-) \end{array}$$

The second is left as an exercise. □

Lemma 4 (Explosion)**.** *The following are 'derived rules' of the system:*

$$A, \neg A \Rightarrow B \qquad A \wedge \neg A \Rightarrow B$$

$$\begin{array}{rr} \Gamma \Rightarrow A & \Gamma \Rightarrow A \wedge \neg A \\ \Delta \Rightarrow \neg A & \therefore \Gamma \Rightarrow B \\ \therefore \Gamma, \Delta \Rightarrow B & \end{array}$$

Proof. For the third:

$$\begin{array}{rr} \Gamma \Rightarrow A & \\ \Gamma, \neg B \Rightarrow A & \text{(thinning)} \\ \Delta \Rightarrow \neg A & \\ \Delta, \neg B \Rightarrow \neg A & \text{(thinning)} \\ \Gamma . \Delta \Rightarrow \neg\neg B & (\neg+) \\ \Gamma . \Delta \Rightarrow B & (\neg-) \end{array}$$

The first then follows by taking Γ to be $\{A\}$ and Δ to be $\{B\}$. The second follows by taking both Γ and Δ to be $\{A \wedge \neg A\}$ and applying Lemma 2. The fourth follows from that by the second form of the cut rule. □

The derived rules for $\bot$ mentioned above now follow from explosion.

Lemma 5 (Disjunctive Syllogism)**.** *The following is a 'derived rule':*

$$A \vee B, \neg A \Rightarrow B$$

Proof.

$$\begin{array}{rr} A, \neg A \Rightarrow B & \\ B \Rightarrow B & \\ A \vee B, \neg A \Rightarrow B & (\vee+) \end{array}$$

□

3.3 The Satisfaction Properties

Lemma 6. *The set Σ of consistent sets of sentences has the satisfaction properties.*

We will here be concerned only with the system as limited to $\neg$, $\vee$, and $\exists$, and it should be noted that we will only be using the rules for these connectives. To extend the result to the other connectives, one can proceed in a couple different ways:

- Regard the other connectives as abbreviations, and show that the introduction and elimination rules for them can be derived from the rules for the other connectives.
- Add conditions to the 'satisfaction properties' for the other connectives that are analogous to the existing conditions, adapt the proof of the Term Models Lemma to include formulae containing those connectives, and show directly that these conditions are satisfied by the set of consistent sets of sentences.

Both of these are worthwhile exercises.

Proof. (S0) If $\Gamma \in \Sigma$ and $\Delta \subseteq \Gamma$, then $\Delta \in \Sigma$. Contraposing, what this says is that, if Δ is *in*consistent, and $\Delta \subseteq \Gamma$, then Γ is inconsistent. This is easy:

$$\begin{array}{ll} \Delta \Rightarrow \bot & \\ \Gamma \Rightarrow \bot & \text{(thinning)} \end{array}$$

(S1) If $\Gamma \in \Sigma$, then for no A are both $A, \neg A \in \Sigma$. Contraposing again, what this says is that, if $A, \neg A \in \Sigma$, then Σ is inconsistent. This, too, is easy:

$$\begin{array}{l} \Gamma \Rightarrow A \\ \Gamma \Rightarrow \neg A \\ \Gamma \Rightarrow \bot \end{array}$$

The first two steps follow from Lemma 2.

(S2) If $\Gamma \in \Sigma$ and $\neg\neg B \in \Gamma$, then $\Gamma \cup \{B\} \in \Sigma$. Let Δ be $\Gamma \setminus \{\neg\neg B\}$. Then, contraposing, this says: If $\Delta \cup \{\neg\neg B, B\}$ is inconsistent, so is $\Delta \cup \{\neg\neg B\}$. In fact, we can show more: If $\Delta \cup \{B\}$ is inconsistent, so is $\Delta \cup \{\neg\neg B\}$.

$$\begin{array}{rll} \Delta, \neg\neg B, B & \Rightarrow \bot & \\ \neg\neg B & \Rightarrow \neg\neg B & \\ \neg\neg B & \Rightarrow B & (\neg\text{-}) \\ \Delta, \neg\neg B & \Rightarrow \bot & \text{(cut)} \end{array}$$

Note that we could have proceeded differently, and shown

(S3) If $\Gamma \in \Sigma$ and $B \vee C \in \Gamma$, then either $\Gamma \cup \{B\} \in \Sigma$ or $\Gamma \cup \{C\} \in \Sigma$. Again, let Δ be $\Gamma \setminus \{B \vee C\}$. Then, contraposing, this says: If both $\Delta \cup \{B \vee C, B\}$ and $\Delta \cup \{B \vee C, C\}$ are inconsistent, then so is $\Delta \cup \{B \vee C\}$.

$$
\begin{array}{rlr}
\Delta, B \vee C, B \Rightarrow \bot & & \\
B \Rightarrow B & & \\
B \Rightarrow B \vee C & & (\vee+) \\
\Delta, B \Rightarrow \bot & & (\text{cut}) \\
\Delta \Rightarrow \neg B & & (\text{Lemma 1}) \\
\Delta, B \vee C, C \Rightarrow \bot & & \\
\Delta \Rightarrow \neg C & & (\text{similar}) \\
B \vee C \Rightarrow B \vee C & & \\
\Delta, B \vee C \Rightarrow B & & (\text{Lemma 5}) \\
\Delta, B \vee C \Rightarrow B \wedge \neg B & & (\wedge+) \\
\Delta, B \vee C \Rightarrow \bot & &
\end{array}
$$

(S4) If $\Gamma \in \Sigma$ and $\neg(B \vee C) \in \Gamma$, then $\Gamma \cup \{\neg B\} \in \Sigma$ and $\Gamma \cup \{\neg C\} \in \Sigma$. Again, let Δ be $\Gamma \setminus \{\neg(B \vee C)\}$. Then, contraposing, this says: If either $\Delta \cup \{\neg(B \vee C), \neg B\}$ or $\Delta \cup \{\neg(B \vee C), \neg C\}$ is inconsistent, then so is $\Delta \cup \{\neg(B \vee C)\}$.

$$
\begin{array}{rlr}
\Delta, \neg(B \vee C), \neg B \Rightarrow \bot & & \\
\Delta, \neg(B \vee C) \Rightarrow \neg\neg B & & (\text{Lemma 1}) \\
\Delta, \neg(B \vee C) \Rightarrow B & & (\neg\text{-}) \\
\Delta, \neg(B \vee C) \Rightarrow B \vee C & & (\vee+) \\
\Delta, \neg(B \vee C) \Rightarrow \bot & & (\text{Lemma 1})
\end{array}
$$

(S5) If $\Gamma \in \Sigma$ and $\exists x A(x) \in \Sigma$, and c is not in Γ, then $\Gamma \cup \{A(c)\} \in \Sigma$. So we must show that if $\Delta \cup \{\exists x A(x), A(c)\}$ is inconsistent, so is $\Delta \cup \{\exists x A(x)\}$.

$$
\begin{array}{rlr}
\Delta, \exists x A(x), A(c) \Rightarrow \bot & & \\
\exists x A(x) \Rightarrow \exists x A(x) & & \\
\Delta, \exists x A(x) \Rightarrow \bot & & (\exists\text{-})
\end{array}
$$

(S6) If $\Gamma \in \Sigma$ and $\neg\exists x A(x) \in \Sigma$, then $\Gamma \cup \{\neg A(t)\} \in \Sigma$, for all t. So we must show that if $\Delta \cup \{\neg\exists x A(x), \neg A(t)\}$ is inconsistent, for any t, so is $\Delta \cup \{\neg\exists x A(x)\}$.

$$
\begin{array}{rlr}
\Delta, \neg\exists x A(x), \neg A(t) \Rightarrow \bot & & \\
\Delta, \neg\exists x A(x) \Rightarrow A(t) & & (\text{Lemma 1}) \\
\Delta, \neg\exists x A(x) \Rightarrow \exists x A(x) & & (\exists+) \\
\Delta, \neg\exists x A(x) \Rightarrow \bot & & (\text{Lemma 1})
\end{array}
$$

(S7) If $\Gamma \in \Sigma$, then $\Gamma \cup \{t = t\} \in \Sigma$. Contraposing, we want to show that if $\Gamma \cup \{t = t\}$ is inconsistent, then so is Γ.

$$
\begin{aligned}
\Gamma, t = t &\Rightarrow \bot \\
\emptyset &\Rightarrow t = t && \text{(=+)} \\
\Gamma &\Rightarrow \bot && \text{(cut)}
\end{aligned}
$$

(S8) If $\Gamma \in \Sigma$, and $B(s), s = t \in \Gamma$, then $\Gamma \cup \{B(t\}) \in \Sigma$. Contraposing, if $\Delta \cup \{B(s), s = t, B(t)\}$ is inconsistent, then so is $\Delta \cup \{B(s), s = t\}$.

$$
\begin{aligned}
\Delta, B(s), s = t, B(t) &\Rightarrow \bot \\
B(s) &\Rightarrow B(s) \\
s = t &\Rightarrow s = t \\
B(s), s = t &\Rightarrow B(t) && \text{(=-)} \\
\Delta, B(s), s = t &\Rightarrow \bot && \text{(cut)}
\end{aligned}
$$

That completes the proof. □