The Closure Lemma

Lemma (13.6, Closure Lemma)**.** *Let $\mathcal{L}$ be a language, and let $\mathcal{L}^+$ be the result of adding infinitely many new constants $d_0, d_1, \ldots$ to $\mathcal{L}$. If Σ is a family of sets of $\mathcal{L}^+$ sentences that has the satisfaction properties, then every set $\Gamma \in \Sigma$ of $\mathcal{L}$-sentences can be extended to a set Γ^* that has the closure properties.*

The proof of this result illustrates a very common sort of construction, one that is made possible because the language with which we are dealing is enumerable. Similar constructions can be carried out for non-enumerable languages, but in that case one needs to appeal to the so-called Axiom of Choice.

Proof. Let $\phi_1, \phi_2, \ldots$ be an enumeration of all the sentences of $\mathcal{L}^+$. We now construct a sequence of extensions of Γ whose union (limit, in a sense) is the wanted set Γ^*. The construction is described formally in Figure 1 on page 2.

We start with our set Γ. We then proceed to go through the sentences ϕ_n. At each stage, we first check to see if adding ϕ_n to Γ_{n-1} would give us a larger set that is still in Σ. If so, we add it; if not, we leave it out. This gives us Γ'_n. Then, *if* ϕ_n was an existential sentence, and *if* we did in fact add it, then we also add a 'witness': an instance of that existential statement, using the first constant d_m that does not already appear in Γ'_n. (Obviously, if we did not add ϕ_n, we do not want to add a 'witness'.)

Note that this guarantees at the Γ_n are non-decreasing. I.e., $\Gamma_m \subseteq \Gamma_n$ if $m \leq n$. Note, moreover, that $\Gamma'_n \in \Sigma$, by construction. Hence $\Gamma_n \in \Sigma$, by (S5). So $\Gamma_n \in \Sigma$, for all n.

We now need to see that Γ^* really does have the closure properties.

For (C1), suppose $A \in \Gamma^*$ and suppose further that $\neg A \in \Gamma^*$. Then for some i, j, $A \in \Gamma_i$ and $\neg A \in \Gamma_j$. Let $k = max(i, j)$. then $A \in \Gamma_k$ and $\neg A \in \Gamma_k$. But $\Gamma_k \in \Sigma$, and that violates condition (S1).

For (C2), suppose $\neg\neg B \in \Gamma^*$. So for some n, $\neg\neg B \in \Gamma_n$. Suppose $B = \phi_i$. Then there are two cases to consider.

- $\Gamma_0 = \Gamma$
- $\Gamma'_1 = \begin{cases} \Gamma_0 \cup \{\phi_1\} & \text{if } \Gamma_0 \cup \{\phi_1\} \in \Sigma \\ \Gamma_0 & \text{otherwise} \end{cases}$
- $\Gamma_1 = \begin{cases} \Gamma'_1 \cup \{B(d_k)\} & \text{if } \phi_1 = \exists x B(x) \text{ and } \phi_1 \in \Gamma_1 \text{ and} \\ & k \text{ is the least } m \text{ such that } d_m \text{does not occur} \\ & \text{in } \Gamma_1 \\ \Gamma'_1 & \text{otherwise} \end{cases}$
- ...
- $\Gamma'_n = \begin{cases} \Gamma_{n-1} \cup \{\phi_n\} & \text{if } \Gamma_{n-1} \cup \{\phi_n\} \in \Sigma \\ \Gamma_{n-1} & \text{otherwise} \end{cases}$
- $\Gamma_n = \begin{cases} \Gamma'_n \cup \{B(d_k)\} & \text{if } \phi_n = \exists x B(x) \text{ and } \phi_n \in \Gamma'_n \text{ and} \\ & k \text{ is the least } m \text{ such that } d_m \text{does not occur} \\ & \text{in } \Gamma'_n \\ \Gamma'_n & \text{otherwise} \end{cases}$
- $\Gamma^* = \bigcup_i \Gamma_i$

Fig. 1: The Lindenbaum Construction

1. Suppose $n < i$. Then $\neg\neg B \in \Gamma_{i-1} \in \Sigma$, so $\Gamma_{i-1} \cup \{B\} \in \Sigma$, by (S2). So $B \in \Gamma'_i \subseteq \Gamma^*$.
2. Suppose $i \leq n$. Then $\neg\neg B \in \Gamma_n \in \Sigma$, so $\Gamma_n \cup \{B\} \in \Sigma$. But $\Gamma_{i-1} \cup \{B\} \subseteq \Gamma_n \cup \{B\}$, so $\Gamma_{i-1} \cup \{B\} \in \Sigma$, by (S1), and hence $B \in \Gamma'_i \subseteq \Gamma^*$.

For (C3), suppose $(A \vee B) \in \Gamma^*$. So $(A \vee B) \in \Gamma_n$, A is ϕ_i, and B is ϕ_j, for some n, i, and j. We thus know from (S3) that either $\Gamma_n \cup \{A\} \in \Sigma$ or $\Gamma_n \cup \{B\} \in \Sigma$.

We suppose without loss of generality that $i < j$, and there are then again two cases to consider.

1. Suppose $n < j$. Then $(A \vee B) \in \Gamma_{j-1} \in \Sigma$, and so either $\Gamma_{j-1} \cup \{A\} \in \Sigma$ or $\Gamma_{j-1} \cup \{B\} \in \Sigma$. If the latter, then $B \in \Gamma'_j \subseteq \Gamma^*$. And if the former, then $\Gamma_{i-1} \cup \{A\} \subseteq \Gamma_{j-1} \cup \{A\}$, so $\Gamma_{i-1} \cup \{A\} \in \Sigma$, and $A \in \Gamma'_i \subseteq \Gamma^*$.

2. Suppose $i < j \leq n$. Then either $\Gamma_{n-1} \cup \{A\} \in \Sigma$ or $\Gamma_{n-1} \cup \{B\} \in \Sigma$. Suppose it's the former. Then $\Gamma_{i-1} \cup \{A\} \subseteq \Gamma_{n-1} \cup \{A\}$, so $\Gamma_{i-1} \cup \{A\} \in \Sigma$, and $A \in \Gamma'_i \subseteq \Gamma^*$.

For (C5), suppose $\exists x B(x) \in \Gamma^*$. Now, for some i, $\exists x B(x)$ is ϕ_i, and it is easy to see that, therefore, $\exists x B(x) \in \Gamma_{2i-1}$. So, by construction, for some d_k, $B(d_k) \in \Gamma_{2i} \subseteq \Gamma^*$.

For (C8), suppose $B(s) \in \Gamma$ and $s = t \in \Gamma$. So $B(s) \in \Gamma_m$ and $s = t \in \Gamma_n$. Let k be the greater of the two, so $B(s), s = t \in \Gamma_k$. Now, for some i, $B(t)$ is ϕ_i. There are, as usual, two cases to consider.

1. Suppose $k < i$. Then $B(s), s = t \in \Gamma_{i-1}$, and so $\Gamma_{i-1} \cup \{B(t)\} \in \Sigma$, whence $B(t) \in \Gamma'_i \subseteq \Gamma^*$.

2. Suppose $i \leq k$. Then $\Gamma_k \cup \{B(t)\} \in \Sigma$ and $\Gamma_{i-1} \cup \{B(t)\} \subseteq \Gamma_k \cup \{B(t)\}$, so $\Gamma_{i-1} \cup \{B(t)\} \in \Sigma$ and $B(t) \in \Gamma'_i \subseteq \Gamma^*$, again

(C4), (C6), and (C7) are left as exercises. □