Philosophy 1880
The Sequent Calculus

$A \Rightarrow A$
(Basic Sequents)

If $\Gamma \Rightarrow A$ and $\Delta \supseteq \Gamma$, then $\Delta \Rightarrow A$
(Thinning)

(¬+)
$\Gamma, A \Rightarrow B$
$\Delta, A \Rightarrow \neg B$
$\therefore \Gamma, \Delta \Rightarrow \neg A$

(¬-)
$\Gamma \Rightarrow \neg\neg A$
$\therefore \Gamma \Rightarrow A$

(∨+)
$\Gamma \Rightarrow A$
$\therefore \Gamma \Rightarrow A \vee B$

(∨+)
$\Gamma \Rightarrow B$
$\therefore \Gamma \Rightarrow A \vee B$

(∨-)
$\Gamma, A \Rightarrow C$
$\Delta, B \Rightarrow C$
$\Theta \Rightarrow A \vee B$
$\therefore \Gamma, \Delta, \Theta \Rightarrow C$

(∧+)
$\Gamma \Rightarrow A$
$\Delta \Rightarrow B$
$\Gamma, \Delta \Rightarrow A \wedge B$

(∧-)
$\Gamma \Rightarrow A \wedge B$
$\therefore \Gamma \Rightarrow A$

(∧-)
$\Gamma \Rightarrow A \wedge B$
$\therefore \Gamma \Rightarrow B$

(→+)
$\Gamma, A \Rightarrow B$
$\therefore \Gamma \Rightarrow A \to B$

(→-)
$\Gamma \Rightarrow A \to B$
$\Delta \Rightarrow A$
$\therefore \Gamma, \Delta \Rightarrow B$

(=+)
$\emptyset \Rightarrow t = t$

(=-)
$\Gamma \Rightarrow A(s)$
$\Delta \Rightarrow s = t$
$\therefore \Gamma, \Delta \Rightarrow A(t)$

$$\frac{\Gamma \Rightarrow A(t)}{\Gamma \Rightarrow \exists x A(x)} \quad (\exists+)$$

$$\Gamma, A(y) \Rightarrow B \quad (\exists\text{-})$$
$$\Delta \Rightarrow \exists x A(x)$$
$$\therefore \Gamma, \Delta \Rightarrow B$$

$$\Gamma \Rightarrow A(y) \quad (\forall+)$$
$$\therefore \Gamma \Rightarrow \forall x A(x)$$

$$\frac{\Gamma \Rightarrow \forall x A(x)}{\Gamma \Rightarrow A(t)} \quad (\forall\text{-})$$

Note: In the identity and quantifier rules, t and s can be any term, subject to restrictions on capturing; y can be any variable, subject to the condition, in $\exists-$, that it not be free in Γ or in B, and in $\forall+$, that it not be free in Γ.

Example Deductions

$\forall x(Fx \to Gx), \exists x(Fx) \Rightarrow \exists x(Gx)$

(1)	$\forall x(Fx \to Gx) \Rightarrow \forall x(Fx \to Gx)$	Basic
(2)	$\forall x(Fx \to Gx) \Rightarrow Fa \to Ga$	$\forall-$
(3)	$Fa \Rightarrow Fa$	Basic
(4)	$\forall x(Fx \to Gx), Fa \Rightarrow Ga$	(2,3)$\to -$
(5)	$\forall x(Fx \to Gx), Fa \Rightarrow \exists x(Gx)$	(4)$\exists+$
(6)	$\exists x(Fx) \Rightarrow \exists x(Fx)$	Basic
(7)	$\forall x(Fx \to Gx), \exists x(Fx) \Rightarrow \exists x(Gx)$	(5,6)$\exists-$

$\exists y \forall x(Lxy) \Rightarrow \forall x \exists y(Lxy)$

(1)	$\forall x(Lxa) \Rightarrow \forall x(Lxa)$	Basic
(2)	$\forall x(Lxa) \Rightarrow Lxa$	(1)$\forall-$
(3)	$\forall x(Lxa) \Rightarrow \exists y(Lxy)$	(2)$\exists+$
(4)	$\forall x(Lxa) \Rightarrow \forall x \exists y(Lxy)$	(3)$\forall+$
(5)	$\exists y \forall x(Lxy) \Rightarrow \exists y \forall x(Lxy)$	Basic
(6)	$\exists y \forall x(Lxy) \Rightarrow \forall x \exists y(Lxy)$	(4,5)$\exists-$

And now for a more complicated example.

$\forall x(\exists y(Ayx) \to Axx), \forall x \forall y(Lxy \to Axy) \Rightarrow \forall x(\exists y(Lyx) \to \exists y(Ayx))$

(1)	$Lau \Rightarrow Lau$	Basic
(2)	$\forall x \forall y(Lxy \rightarrow Axy) \Rightarrow \forall x(Lxy \rightarrow Axy)$	Basic
(3)	$\forall x \forall y(Lxy \rightarrow Axy) \Rightarrow \forall y(Lay \rightarrow Aay)$	(2)$\forall -$
(4)	$\forall x \forall y(Lxy \rightarrow Axy) \Rightarrow Lau \rightarrow Aau$	(3)$\forall -$
(5)	$\forall x \forall y(Lxy \rightarrow Axy), Lau \Rightarrow Aau$	(1,4)$\rightarrow -$
(6)	$\exists y(Lyu) \Rightarrow \exists y(Lyu)$	Basic
(7)	$\forall x \forall y(Lxy \rightarrow Axy), \exists y(Lyu) \Rightarrow Aau$	(5,6)$\exists -$
(8)	$\forall x \forall y(Lxy \rightarrow Axy), \exists y(Lyu) \Rightarrow \exists u(Ayu)$	(7)$\exists +$
(9)	$\forall x(\exists y(Ayx) \rightarrow Axx) \Rightarrow \forall x(\exists y(Ayx) \rightarrow Axx)$	Basic
(10)	$\forall x(\exists y(Ayx) \rightarrow Axx) \Rightarrow \exists y(Ayu) \rightarrow Auu$	(9)$\forall -$
(11)	$\forall x(\exists y(Ayx) \rightarrow Axx), \forall x \forall y(Lxy \rightarrow Axy), \exists y(Lyu) \Rightarrow Auu$	(8,10)$\rightarrow -$
(12)	$\forall x(\exists y(Ayx) \rightarrow Axx), \forall x \forall y(Lxy \rightarrow Axy), \exists y(Lyu) \Rightarrow \exists y(Ayu))$	(11)$\exists +$
(13)	$\forall x(\exists y(Ayx) \rightarrow Axx), \forall x \forall y(Lxy \rightarrow Axy) \Rightarrow \exists y(Lyu) \rightarrow \exists y(Ayu))$	(12)$\rightarrow +$
(14)	$\forall x(\exists y(Ayx) \rightarrow Axx), \forall x \forall y(Lxy \rightarrow Axy) \Rightarrow \forall x(\exists y(Lyx) \rightarrow \exists y(Ayx))$	(13)$\forall +$