‘Theories’ and Related Notions

- A *theory* is a set of sentences from some fixed language $\mathscr{L}$.
- An *axiomatic* (or *formal*) *theory* is a recursive set of sentences: the theory’s axioms.
- The *theorems* of a theory are the sentences deducible from it.
- A *closed theory* is a set of sentences that also contains all its theorems. I.e., Γ is closed if, whenever $\Gamma \vdash A$, then $A \in \Gamma$.
- The *closure* of a theory is the set of its theorems. We write $\mathsf{Cl}(\Gamma)$ for the closure of Γ.
- A theory Γ is *complete* if, for every sentence (not formula!) A in the language of that theory, either A is a theorem of Γ or $\neg A$ is a theorem of Γ.
- A theory Γ is *decidable* if its closure is recursive, i.e., if (the coded version of) the relation “A is a theorem of Γ” is recursive.

What the book calls a theory is what I have here called a *closed* theory. Both sorts of terminology are used, but I prefer this terminology for reasons I won’t try to explain now.

As a result, some of the problems for Chapter 15 need to be re-stated.

Problem (15.2)**.** Show that $\mathsf{Cl}(\Gamma)$, the closure of Γ, is closed, for any Γ. (Extra credit: Show that, if Δ is a closed theory and $\Gamma \subseteq \Delta$, then $\mathsf{Cl}(\Gamma) \subseteq \Delta$. I.e., $\mathsf{Cl}(\Gamma)$ is the smallest closed theory containing Γ.)

Problem (15.5)**.** Let $\Gamma = \{A_1, A_2, \ldots\}$ be a countable set of sentences. Suppose that for no n is A_n derivable from $\{A_1, \ldots, A_{n-1}\}$. Show that there is no finite set Δ, whether it is a subset of Γ or not, that has the same theorems as Γ, i.e., such that $\mathsf{Cl}(\Gamma) = \mathsf{Cl}(\Delta)$.