# Theories of Arithmetic

**Definition.** The theory **R** is the infinitely axiomatized theory whose axioms are all instances of the following:

**(R1)** $\overline{n}+\overline{m}=\overline{n+m}$

**(R2)** $\overline{n}\times\overline{m}=\overline{n\times m}$

**(R3)** $\overline{n}\neq\overline{m}$, whenever $n\neq m$

**(R4)** $x<\overline{n}\equiv x=\overline{0}\vee\cdots\vee x=\overline{n-1}$

**(R5)** $x<\overline{n}\vee x=\overline{n}\vee\overline{n}<x$

Note that, for (R4), the right-hand side vanishes if $n=0$, so what is intended is: $\neg x<\overline{0}$.

**Lemma.** ***R*** *proves every true equation and inequation.*

**Lemma.** ***R*** *proves every true atomic, or negated atomic, sentence.*

**Lemma.** ***R*** *proves every true $\Delta_0$ sentence.*

**Theorem** (~16.13)**.** *Every true $\Sigma_1$ sentence is provable in* ***R****.*

**Lemma** (16.14)**.** *Every $\Delta_0$-definable function is representable in* ***R****, and by a $\Delta_0$ formula.*

**Theorem** (16.16)**.** *Every recursive function is representable in* ***R*** *by a generalized $\Sigma_1$ formula. Hence every recursive function is representable by a generalized $\Sigma_1$ formula in every extension of* ***R****, as well.*

By a 'generalized' $\Sigma_1$ formula, we mean one of the form: $\exists x_1\cdots\exists x_n\phi$, where $\phi$ is $\Delta_0$. I.e., we allow multiple existential quantifiers at the front.

**Corollary.** *Every recursive relation (and function) is definable in* ***R*** *by a generalized $\Sigma_1$ formula.*

**Definition.** **Q** is the theory whose axioms are the universal closures of the following eight principles:

**Q1** $Sx \neq 0$

**Q2** $Sx \wedge Sy \rightarrow x = y$

**Q3** $x + 0 = x$

**Q4** $x + Sy = S(x + y)$

**Q5** $x \times 0 = 0$

**Q6** $x \times Sy = (x \times y) + x$

**Q7** $x \neq 0 \rightarrow \exists y(x = Sy)$

**Q8** $x < y \equiv \exists z(y = Sz + x)$

The last axiom is sometimes regarded as a definition of $<$, but we shall treat it as an axiom. Note that **Q** is finitely axiomatized: It has exactly eight axioms.

**Theorem.** ***Q*** *contains* ***R****. That is: Every axiom of* ***R*** *is a theorem of* ***Q****, and so of every theory that contains* ***Q****.*

**Corollary.** *Every recursive function is representable in* ***Q*** *by a* $\Sigma_1$ *formula.*

**Definition.** Peano arithmetic, or **PA**, is the infinitely axiomatized theory whose axioms are (Q1)–(Q6), (Q8), and all instances of the induction scheme:

$$A(0) \wedge \forall x(A(x) \rightarrow A(Sx)) \rightarrow \forall x(A(x))$$

where $A(x)$ is any formula of the language of arithmetic (subject to the usual sorts of restrictions).

**Fact.** **PA** contains **Q**. I.e., **PA** proves (Q7).