EXERCISES

Exercise 2.1. Gödel does not include '=' in the language of his theory P, because it can be defined in simple type theory as:

$$u_n = v_n \equiv \forall t_{n+1}(t_{n+1}(u_n) \to t_{n+1}(v_n))$$

The equivalent definition for objectual identity in second-order logic is:

$$a = b \equiv \forall F(Fa \to Fb)$$

You may do the various parts of this exercise using either definition.

(i) Show that this definition is equivalent to the apparently stronger definition

$$a = b \equiv \forall F(Fa \equiv Fb)$$

in which '$\to$' has been replaced by '$\equiv$'.

(ii) Show that the other basic properties of identity

$$a = a$$
$$a = b \to b = a$$
$$a = b \wedge b = c \to a = c$$

follow from the definition.

Your proofs need not be completely formal, but the proofs should be formal-ish, i.e., they should make it clear that the claimed implications are *logical*.

Exercise 2.2. Gödel uses the fact that the relation $x < y$ is recursive, but does not prove it. Prove it.

Hint: First show that

$$m \dot{-} n = \begin{cases} m - n & \text{, if } n < m \\ 0 & \text{, if } m \leq n \end{cases}$$

sometimes called 'monus', is recursive. (It's basically subtraction, but cut off at zero.)

Exercise 2.3. Show that the functions

$$\beta(x, y) = \begin{cases} 1 & \text{, if } x > 0 \wedge y > 0 \\ 0 & \text{, otherwise} \end{cases}$$

and

$$\gamma(x, y) = \begin{cases} 0 & \text{, if } x = y \\ 1 & \text{, if } x \neq y \end{cases}$$

are (primitive) recursive. (Hint: What you did in the previous problem might prove helpful.)

Exercise 2.4. In the proof of Theorem IV, Gödel defines a function

$$\chi(z, \vec{y}) = \begin{cases} 0 & \text{, if } z < n \\ n & \text{, if } n \leq z \end{cases}$$

where n is the least number for which $R(n, \vec{y})$ holds, if any, and χ is constantly 0 if there is no such n. The definition is as follows:

$$\begin{aligned} \chi(0, \vec{y}) &= 0 \\ \chi(n+1, \vec{y}) &= a(n) \cdot (n+1) + \alpha(a(n)) \cdot \chi(n, \vec{y}) \end{aligned}$$

where:

$$a(n) = \alpha(\alpha(\rho(0, \vec{y}))) \cdot \alpha(\rho(n+1, \vec{y})) \cdot \alpha(\chi(n, \vec{y}))$$

$$\alpha(x) = \begin{cases} 0 & \text{, if } x \neq 0 \\ 1 & \text{, if } x = 0 \end{cases}$$

and $\rho(z, \vec{y})$ is the characteristic function of the relation $R(z, \vec{y})$. Explain why the definition of χ works.

Exercise 2.5. Gödel's coding of finite sequences works only for positive integers. Let's explore this situation.

(i) If we try to code the sequence <$2, 0, 2$> Gödel's way, we get $2^2 \cdot 3^0 \cdot 5^2 = 100$. But his definition of $n \,\mathrm{Gl}\, s$ tells us that $1 \,\mathrm{Gl}\, 100 = 2$, $2 \,\mathrm{Gl}\, 100 = 2$, and $3 \,\mathrm{Gl}\, 100 = 0$, which is wrong. Explain why.

(ii) We can fix this problem by slightly changing the coding, so that <$3, 5, 7$> is coded as $2^4 \cdot 3^6 \cdot 5^8$, and <$2, 0, 2$> can then be coded as $2^3 \cdot 3^1 \cdot 5^3$. How would we have to change Gödel's definition (6) to make this work?

(iii) On neither treatment does every number code a sequence. Which numbers do code sequences?

(iv) Show that the property of *being a number that codes a sequence* is (primitive) recursive by defining it in the same sort of way Gödel defines his 45 notions. You may do so for either of the two codings.

(v) Consider the alternate definition:

$$n \,\mathrm{Gl}\, s \equiv \mu y \leq s[s/\mathrm{Pr}(n)^y \wedge \neg(x/\mathrm{Pr}(n)^{y+1}]$$

Show that this also allows us, to some extent, to code sequences containing zero: I.e., $1 \,\mathrm{Gl}\, 100 = 2$, $2 \,\mathrm{Gl}\, 100 = 0$, and $3 \,\mathrm{Gl}\, 100 = 2$. Unfortunately, this coding has a different problem. Which? Can you think of some way to solve that problem?

Exercise 2.6. Gödel's definition of concatenation is a bit magical. Show that

$$\begin{aligned} \mathrm{Cat}(s, t, 0) &= s \\ \mathrm{Cat}(s, t, n+1) &= \mathrm{Cat}(s, t, n) \cdot Pr(l(\mathrm{Cat}(s, t, n)) + 1)^{(n+1) \,\mathrm{Gl}\, t} \\ s * t &= \mathrm{Cat}(s, t, l(t)) \end{aligned}$$

also works, and explain why maybe the new definition isn't so magical.

Exercise 2.7. Definitions (24)–(30) lead up to the crucial definition at (31) of *substitution*. Explain what each one of (25)–(31) means, and why it works. We did (24) in class. You should follow something like my example. Here is my version of that one:

> What (24) says is that an occurrence of the variable v at the n^{th} place in the formula x is *bound* if it is within a subformula of the form $\forall v(\ldots v \ldots)$. (I.e., we are talking about the n^{th} character in the string.) Here, b would correspond to $\ldots v \ldots$; it is what is being generalized. And a and c are just the 'parts' of the formula x that come before and after $\forall v(\ldots v \ldots)$. So the long clause on the second line says that $\forall v(\ldots v \ldots)$, i.e., v Gen b, occurs in the formula x, preceded by a and followed by b—which, note, will not themselves ordinarily be formulas. The last conjunct says that the n^{th} place in the formula is, indeed, within $\forall v(\ldots v \ldots)$. Note, however, that Gödel's definition does not actually require that v *does* occur at the n^{th} place. So what it really says is that v, if it *did* occur there, *would* be bound. We could fix this by adding $v = n \,\mathrm{Gl}\, x$ as another conjunct, but we will actually need the more general notion below, at (37).

Yours need not be quite so long in every case. The really important ones are (27), (28), (30), and (31).

Exercise 2.8. At definition (35), Gödel does not define the formulas A_2-$Ax(x)$, A_3-$Ax(x)$, and A_4-$Ax(x)$. Define them for him.

Exercise 2.9. Say that a set A is 'additive' if it satisfies the following two conditions:

$$<x, 0, x> \in A$$
$$<x, y, z> \in A \rightarrow <x, Sy, Sz> \in A$$

(i) Show that, if $x + y = z$, then $<x, y, z>$ is in every additive set. (Hint: Induction.)

(ii) Show, *using only the recursion equations for addition*, that $\{<x, y, z> : x + y = z\}$ is itself an additive set. (Hint: The two conditions on additive sets correspond to the two recursion equations.)

(iii) Using (i) and (ii), show that $\{<x, y, z> : x+y = z\}$ is the intersection of all the additive sets and so that, if we define

$$\mathrm{sum}(x, y, z) \equiv <x, y, z> \in \bigcap\{A : A \text{ is an additive set}\}$$

then $\mathrm{sum}(x, y, z)$ iff $x + y = z$.

Exercise 2.10. Using the same techniques as in Exercise 2.9:

(i) Show how to define the factorial.

(ii) Show how to define the class of terms of the language of arithmetic.

Give an argument, similar to that in Exercise 2.9, to show that the definition given for the factorial is, in fact, adequate.

Exercise 2.11. In the proof of Theorem VII, Gödel omits the basis case: He doesn't bother to show that the successor function $Sx = x + 1$ and the various constant functions $c_n(x) = n$ are arithmetically definable. Help him out by showing that these two functions are definable in (first-order) logic in terms of the two operations of addition and multiplication. (So, in particular, you cannot use '0' or '1'.) That is, show that there are arithmetical formulas Succ(x, y) and, for each k, C$_k(x, y)$, such that Succ$(\overline{n}, \overline{m})$ is true iff $m = n + 1$ and C$_k(\overline{n}, \overline{m})$ is true iff $m = k$.

Exercise 2.12. Suppose that there are arithmetical formulae $G(x, y, z)$, $F_1(x, y)$, $F_2(x, y)$ that define $g(x, y)$, $f_1(x)$, and $f_2(x)$, respectively. Show that

$$H(x, y) \equiv \exists a \exists b([F_1(x, a) \wedge F_2(x, b) \wedge G(a, b, y)]$$

will then define $h(x) = g(f_1(x), f_2(x))$.

Note: This is not difficult. The point is to get all the details right. Be especially careful about when you are talking about numbers and when you are talking about numerals!