Exercises on Tarski, Mostowski, and Robinson

Exercise 3.1. Prove, as carefully as possible, and by appeal to Tarski's definition of interpretability, that interpretability is reflexive and transitive. I.e.:

(i) Every theory is interpretable in itself.

(ii) If S is interpretable in T and T is interpretable in U, then S is interpretable in U.

With respect to (i), make sure you note the remarks Tarski makes in the middle of p. 21.

Exercise 3.2. A theory T is said to be *locally* (relatively) interpretable in a theory B if every finite sub-theory of T is (relatively) interpretable in B. (Obviously, this notion is useful only if T is infinite.) Show that, if T is locally (relatively) interpretable in B, then T is consistent if B is.

Note: When we are potentially interested in local interpretability, it is common to speak of 'global' interpretability, meaning interpretability in the usual sense. And yes: There are cases in which one theory is locally interpretable in another, but not globally interpretable.

Exercise 3.3. Tarski, Mostowski, and Robinson claim on p. 56 that, if there is a formula $F(x_1, \ldots, x_n, y)$ that meets conditions (i) and (ii) in their definition of representability, then the formula

$$F(x, y) \land \forall z[F(x, z) \to y \leq z]$$

will meet all three of the conditions. Show that this is true. i.e., that if:

(i) Whenever $f(k) = m$, $\mathcal{T}$ proves $F(\overline{k}, \overline{m})$.

(ii) Whenever $f(k) \neq m$, $\mathcal{T}$ proves $\neg F(\overline{k}, \overline{m})$.

then also:

(i) Whenever $f(k) = m$, $\mathcal{T}$ proves $F(\overline{k}, \overline{m}) \land \forall z[F(\overline{k}, z) \to \overline{m} \leq z]$.

(ii) Whenever $f(k) \neq m$, $\mathcal{T}$ proves $\neg\{F(\overline{k}, \overline{m}) \land \forall z[F(\overline{k}, z) \to \overline{m} \leq z]\}$.

(iii) For each k, $\mathcal{T}$ proves $\{F(\overline{k}, x) \land \forall z[F(\overline{k}, z) \to x \leq z]\} \land \{F(\overline{k}, y) \land \forall z[F(\overline{k}, z) \to y \leq z]\} \to x = y$.

You may assume that f is a total function (i.e., that it is always defined).

Is it enough here to assume that $\mathcal{T}$ contains R? Or do we have to assume something stronger about $\mathcal{T}$? (Hint: Since f is total, for each k, there is some m such that $\mathcal{T}$ proves $F(\overline{k}, \overline{m})$.)

Exercise 3.4. As Tarski, Mostowski, and Robinson state the theory R, scheme (R4) takes the conditional form:

$$x \leq \overline{n} \to x = \overline{0} \lor \cdots \lor x = \overline{n}$$

Show that R in fact proves all instances of the stronger scheme:

$$x \leq \overline{n} \equiv x = \overline{0} \lor \cdots \lor x = \overline{n}$$

Hint: The proof will have to appeal to the definition of $x \leq y$ as $\exists z(z + x = y)$.

Exercise 3.5. (Somewhat Challenging) Consider the theory R_0 whose axioms are all instances of (R1)–(R4) and in which $\leq$ is defined via $x \leq y \overset{df}{\equiv} \exists z(z + x = y)$. It was observed by Alan Cobham that R_0 is essentially undecidable. More precisely, what he showed in that R_0 interprets R (with $\leq$ treated as primitive), which we showed above is itself essentially undecidable.

Prove Cobham's result by showing that, if we use the definition:

$$x \preceq y \stackrel{df}{\equiv} \{0 \leq y \land \forall u(u \leq y \land u \neq y \to Su \leq y) \to x \leq y\}$$

where $\preceq$ is the 'new' relation we are defining and $\leq$ is the 'old' one from R_0, then R_0 proves the translations of (R4) and (R5), these being:

(R4*) $$\{0 \leq \overline{n} \land \forall u(u \leq \overline{n} \land u \neq \overline{n} \to Su \leq \overline{n}) \to x \leq \overline{n}\} \to x = \overline{0} \lor \cdots \lor x = \overline{n}$$

(R5*) $$\{0 \leq \overline{n} \land \forall u(u \leq \overline{n} \land u \neq \overline{n} \to Su \leq \overline{n}) \to x \leq \overline{n}\} \lor \{0 \leq x \land \forall u(u \leq x \land u \neq x \to Su \leq x) \to \overline{n} \leq x\}$$

The redefinition of $\leq$ does not, of course, affect (R1)–(R3), so you do not need to worry about them. (I.e., you can assume that we translate 0, S, $+$, and $\times$ by themselves, so the translation changes nothing unless $\leq$ is involved.)

Exercise 3.6. Show that none of the axioms of Q are provable in R by constructing, for each axiom of Q, a model in which all axioms of R are true but in which that axiom of Q fails.

To do this, you will need to do the following:

(i) Expand the usual model, consisting just of the natural numbers, by adding one or more 'non-standard' (rogue) elements.
(ii) Define how the non-standard elements behave with respect to S, $+$, and $\times$. Note that your definitions of these functions must be *total*: E.g., $x + y$ must be defined for every pair of elements from your domain.
(iii) Show that (R1)–(R5) still hold but that various axioms of Q do not.

There is no problem about keeping (R1)–(R3) true: Since these do not say anything about the non-standard elements, how S, $+$, and $\times$ operate on the non-standard elements cannot affect (R1)–(R3). But it is not trivial to keep (R4) and (R5) true in the model, since these contain *variables* and so do say something about the non-standard elements. More importantly, you cannot just stipulate how the non-standard elements behave with respect to $\leq$, since it is defined in terms of S and $+$. How you define S and $+$ on the non-standard elements will *fix* how $\leq$ behaves with respect to them. So you have to be careful that the way you define S and $+$ does not invalidate (R4) or (R5).

It is actually possible to find a *single* model in which all axioms of R are true but in which *none* of the axioms of Q are true, thus showing that even the *disjunction* of the axioms of Q is not provable in R. You do not need to find a single such model, but you *certainly* should not need to create seven models: Any model that violates one of Q's axioms will likely violate others.

Exercise 3.7. Tarski, Mostowski, and Robinson claim, but do not prove, that R is not finitely axiomatizable. Show that it is not by proving the following even stronger claim: Let S be any sub-theory of R that contains as axioms all instances of (R1)–(R4), but only finitely many instances of (R5). Then there are instances of (R5) that are not provable in S.

To do this, you will need to construct a model in which (R1)–(R4) and the finitely many instances of (R5) are true, but in which at least one other instance is not. See Exercise 3.6 for some reminders about how to do this.

Hint: It suffices to consider the case in which we have all the instances of (R5) up to and including those for $\overline{n}$, but no more. Why?

Exercise 3.8. (Challenging) Can we prove the analogue of Exercise 3.7 for (R4)?

Exercise 3.9. Show that R is 'locally finitely satisfiable', meaning: For every finite subset of the axioms of R, there is a *finite* model that makes them all true.

To define such a model, you must:

(i) Say what its finite domain is.
(ii) Define how S, $+$, and $\times$ work on this finite domain, remembering that these must all be *total* functions.
(iii) Show that whichever finitely many axioms of R are involved here all true in this model.

This provides another proof that R is not finitely axiomatizable. Why?

Hint: It suffices to consider the case in which we have all the instances of (R1)–(R5) involving numerals up to and including $\overline{n}$, but no more. Why?

Exercise 3.10. Show, as Tarski, Mostowski, and Robinson claim in the proof of Theorem 6 of Chapter 2, that, if $\Phi(u, x)$ and $\Psi(u, x)$ represent $G(u)$ and $H(u)$, respectively, then:

(i) The formula $\exists x \exists y(v = x + y \wedge \Phi(u, x) \wedge \Psi(u, y)]$ represents $G(u) + H(u)$.
(ii) The formula $\exists z(\Phi(u, z) \wedge \Psi(z, v))$ represents $G(H(u))$.