Exercises on Boolos

Exercise 4.1. Assuming that the set of Σ_1 sentences is closed under disjunction, conjunction, existential quantification, and bounded universal quantification, show that the set of Δ_1 sentences is closed under disjunction, conjunction, negation, and bounded existential quantification and universal quantification.

Exercise 4.2. Show that all Σ_1 terms are actually Δ_1. That is: If $F(\vec{x}, y)$ is Σ_1 and PA proves

$$\forall \vec{x}[\exists y(F(\vec{x}, y) \wedge \forall z(F(\vec{x}, z) \rightarrow y = z))]$$

then $\exists y(F(\vec{x}, y) \wedge A(y))$ is PA-provably equivalent to $\forall y(F(\vec{x}, y) \rightarrow A(y))$.

Exercise 4.3. Prove the following two claims as carefully as possible.

(i) Show that, if $F(x_1, \ldots, x_n)$ is Δ_1, then, for any $k_1, \ldots, k_n$, either $F(\overline{k}_1, \ldots, \overline{k}_n)$ is provable in PA or else $\neg F(\overline{k}_1, \ldots, \overline{k}_n)$ is provable in PA.

(ii) Let $F(\vec{x}, y)$ be a formula that PA proves to be Σ_1 pterm, and suppose it defines the function $f(\vec{x})$. Show that $F(\vec{x}, y)$ therefore *represents* $f(\vec{x})$ in PA.

Exercise 4.4. Show that, as Boolos claims, (46) on p. 38 is provable in PA.

Exercise 4.5. Explain what the point of (51) and (52) are, and fill in the details of the proof of (51), which Boolos merely sketches.

Exercise 4.6. Consider the case of a two-place function defined primitive recursively via:

$$h(x, 0) = f(x)$$
$$h(x, n+1) = g(x, n, h(x, n))$$

Show, as Boolos claims on p. 40, that

$$\exists s[\mathrm{lh}(s) = sn \wedge F(x, s_0) \wedge \forall m < n(G(x, m, s_m, s_{m+1})) \wedge s_n = y]$$

is a pterm if both F and G are (i.e., that PA proves existence and uniqueness for the displayed formula). You may 'reason in PA'. The proof is by induction on n.

Exercise 4.7. In the proof of "Provable Σ_1 completeness", on pp. 46–8, Boolos leaves many cases unproved.

(i) In the basis of the induction, on p. 47, Boolos proves only the case for addition. Do either $u = v$ or $0 = u$ and either $\mathrm{s}u = v$ or $u \times v = w$.

(ii) In the inductive part of the proof, on p. 48, Boolos does not do the case of disjunction. Do it.

Exercise 4.8. Show that PA $+ \neg$Con(PA) proves its own inconsistency. (It is of course trivial that it proves $\neg$Con(PA), but what we need to show is that it proves $\neg$Con(PA $+ \neg$Con(PA)).)