Reading Notes for Gödel's "Undecidable Propositions"

There are a few aspects of Gödel's notation that are now somewhat non-standard. (There wasn't really much of a standard at all when Gödel was writing.)

- Gödel uses f as we have been using S. So $fff0$ is Gödel's numeral for 3.
- The biggest notational difference is that Gödel uses Π as the universal qunatifier, rather than $\forall$, and he writes the variable that the quantifer binds *before* it rather than after it. So what we would write as: $\forall xFx$, Gödel writes as: $x\Pi Fx$.
- The existential quantifier is written: $(Ex)A(x)$, and it is officially defined in terms of the universal quantifer, in the usual way.

In §1, Gödel gives a very informal argument for the incompleteness theorem. It is well worth trying to translate this argument into the sort of notation we have been using. The striking feature about this arugment is that it appeals to the notion of *truth* in a way the argument we considered earlier does not. As Gödel notes, a large part of his purpose in going through the details in the rest of the paper is to show that the notion of truth need not be involved in the argument. But that does not make the argument given here uninteresting.

The system P that Gödel describes in §2 is a version of what is known as 'simple type theory'. In addition to so-called 'individual variables' (which he calls variables of type 1), and which are supposed to range over numbers, there are also variables of type 2 ranging over sets of numbers; variables of type 3 ranging over sets of sets of numbers; and so forth. One might think of the atomic formulas, then, as being of the form $x_n \in y_{n+1}$, where the subscripts indicate the 'type' of the variable (or term). But Gödel writes the atomic formulas instead as $y_{n+1}(x_n)$, reflecting an older conception on which, say, type 2 variables range over *properties* of numbers.

These 'higher type' variables are used to define addition and multiplication in a way Gödel does not describe here but that we will explore later. More generally, they allow one to capture so-called inductive definitions within P. That is really the only role they play. In practice, then, Gödel only quantifies over numbers, and the higher type variables play no significant role in the paper. Indeed, the main point of section 3 of the paper is to show that higher types need play no role whatsoever.

Some remarks on the axioms of P:

- Axioms I(1) and I(2) are the usual successor axioms.

- Axiom I(3) is a version of induction.

- The axioms in group II are axioms for propositional logic.

- Axiom III(1) we might write as: $\forall v A(v) \rightarrow A(c)$, for any term c (subject to 'capturing' restrictions he formulates carefully). So this is universal instantiation. Gödel uses his 'Subst' notation to make it a bit more explicit that c is being substituted for all free occurences of v.

- Axiom III(2) has the same import as what we might write as: $\forall v(p \rightarrow A(v)) \rightarrow (p \rightarrow \forall v A(v))$. Since Gödel take $\vee$ to be primitive and $\supset$ to be defined, he formulates III(2) in terms of $\vee$

- Axiom IV is what is known as comprehension, and it only applies to higher type objects (sets, sets of sets, etc). We might write it as follows:

  $$\exists u_{n+1} \forall v_n [u_{n+1}(v_n) \equiv A(v_n)]$$

  where $A(u_n)$ is any formula that does not contain u_{n+1} free (and that normally will contain v_n free). So, if you consider a case in which $A(v_n)$ contains *just* v_n free, then what axiom IV tells us is that $A(v_n)$ defines an object of type $n+1$, i.e., a set of type n objects, namely, those that satisfy $A(v_n)$: The thing u_{n+1} that is asserted to exist applies to exactly those things of which $A(v_n)$ is true.

- Axiom V is an axiom of extensionality, which is clearer if it is written as:

  $$\forall x_1(x_1 \in y_2 \equiv x_1 \in z_2) \rightarrow y_2 = z_2$$

So axiom V says that any ‘two’ type 2 objects that apply to exactly the same things are, actually, the same.

- The system has just two rules of inference: *modus ponens* and universal generalization, as Gödel says right after he finishes introducing the axioms.

What Gödel calls *recursive* functions are what are now called *primitive recursive* functions.