PHIL 1885: Incompleteness

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Course Description

In 1931, the great German logician Kurt Gödel published his paper "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I", which contained Gödel's celebrated incompleteness theorems. The first of these states that, in any consistent axiomatic theory of sufficient strength and expressive power, there will always be a statement that is undecidable by the theory, in the sense that the theory neither proves nor refutes this statement. The second incompleteness theorem states that no consistent axiomatic theory of sufficient strength and expressive power proves its own consistency.

These two results, and the techniques used in their proof, have had a major influence on the development of mathematical logic. Part of our purpose in this course will be simply to understand them.

The first incompleteness theorem underwent significant refinement in the years following Gödel's original publication. What is now its standard treatment did not really emerge until work by Tarski, Mostowski, and Robinson in the early 1950s. We will read their book, in part as a way of becoming familiar with the important notion of 'interpretation'. (Note that this is a different notion from the semantic notion of interpretation that is introduced in Phil 0540.)

Our main focus, though, will be on the second incompleteness theorem: the one concerned with consistency. This result is, in some ways, puzzling. For it turns out that it is extremely sensitive to questions about how notions like *proof* and *theory* are formulated. There are ways of defining these notions that are extensionally correct, in the sense that they get the set of proofs and theorems right, but that seem to allow theories to prove their own consistency, which they are not supposed to be able to do. We will look at some early work on this issue and then, if we have time, at more recent work that seeks to refine our understanding of the insight the second incompleteness theorem seems to give us.

Course Organization

The course will meet at 11am, Monday, Wednesday, and Friday, in Smith-Buonanno 201. I will lecture a fair bit, as there will be material in our readings I need to explain, and material not in our readings I'd like to introduce. It is my hope, however, that there will be a good bit of discussion during our meetings, as well. It will be particularly important, therefore, that everyone come to class prepared not just to listen but to participate, every time, and of course that means reading in advance the material we will be discussing. I will talk some the first day about how to read mathematics.

Prerequisites

Formally speaking, Philosophy 0540, 1630, or 1880 is a prerequisite for this course, though it is possible successfully to take the course without having satisfied the prerequisite, if one's math background is strong enough. Several students do this each time this course is offered.

I will be presuming a familiarity with basic logical notation, with how it can be used to represent the 'logical forms' of ordinary English statements and of mathematical claims, and with basic facts about validity, implication, and formal deduction. So you should understand what something like $\forall x \exists y(Fxy)$ might mean, and how it would differ from $\exists y \forall x(Fxy)$. And you should understand what it means to say that the latter implies the former, but not conversely, and how this could be shown.

Though helpful, Philosophy 1880 is *not* required. While there is some overlap between the material covered in this course and the material covered in that one, the way we approach the common material will be very different, and I will explain all the concepts from 1880 that we need in full. That said, it might be a good idea to have a copy of the text from that course, *Computability and Logic*, by Boolos, Burgess, and Jeffrey, to use as a reference for some of what we shall be reading.

What will be most important, though, is that students should have a degree of mathematical sophistication. The course will be very mathematical in content. It is absolutely essential that students have a solid understanding of what it is to prove something mathematically.

Readings

The only required book for the course is *Undecidable Theories*, by Tarski, Mostowski, and Robinson. I did not order it through the bookstore, as it is readily available from the usual outlets. You may also wish to purchase a copy of *The Undecidable*, edited by Martin Davis. This has Gödel's paper in it (and several other things you really should read, though we will not be reading them, such as Turing's original paper on computability). Note that both of these books are Dover editions, so they are quite cheap.

Other readings will be distributed electronically. Some of these are available online through Brown's digital journal holdings; others will be scans of articles, or chapters from books, that are not otherwise available digitally.

To view PDFs, you will of course need a PDF reader. For the DjVu files, you will need a DjVu reader. Free browser plugins for Windows and Mac OSX are available from Caminova; Linux users can likely just install the djviewlibre package using their distro's package management system. Another option is Okular, which was originally written for Linux's KDE Desktop Environment but which can now be run, experimentally, on Windows and OSX, as well. A list of other DjVu resources is maintained at djvu.org.

The program I use to convert PDFs to DjVu is a simple Bash script I wrote myself, pdf2djvu. It relies upon other programs to do the real work and should run on OSX as well as on Linux.

Requirements

There will be seven problem sets, connected with our readings, due about every couple weeks. There will also be a final exam during the final exam period, possibly take-home.

The exercises will mostly involve filling in the details of proofs from the readings or proving results that are related or similar to ones we discuss in class. As I will emphasize as we go, however, students should really be doing more of this kind of work than will be assigned. It is impossible to learn mathematics without *doing* mathematics, and in this case doing very often means: working out details the author leaves unstated, proving results for which the author does not give a proof, and so forth.

Students are encouraged to work on the problems together, though submitted material should be one's own work. That means you should feel free to get together to discuss the problem sets with others, but your answers should be your own. Do not come up with a "group answer" and then each submit it. Indeed, I'd suggest you not come up with a group answer in the first place. If you do, destroy it, and then re-do the problem by yourself. This is what you need to be able to do, anyway.

You are welcome to do your problem sets by hand or on a computer. But if you are going to do the latter, then I would strongly recommend that you not use a traditional 'word processor' to do so. They are simply not optimized for mathematics, and their output is awful. A much better option is LaTeX, and if you want to use LaTeX in an environment that feels a lot like a word processor, then you can use LyX, which can be downloaded for free from http://lyx.org/. (I am one of the lead developers of LyX, so you should feel free to ask me any questions you may have about it.) Especially if you have any intention of ever doing serious technical writing, you should start using LaTeX (or LyX) sooner rather than later. In the sciences, especially, it is the standard tool. Many scientific journals do not accept submissions in any other form.

Grading Policies

Final grades will be determined by a variety of factors.

- The first and most important factor is that ***all of the problem sets must be completed and submitted for marking***. Failure to submit all of the problem sets will automatically lead to a grade of NC. Please note that the requirement is that the problem sets should be "completed", and by that I mean that one has given them a proper effort. Simply turning in a piece of paper with a few random jottings does not count as completing a problem set.

- The final grade will be determined by the grade on the final and performance on the problem sets, with about half of the grade depending upon each. Borderline cases will be decided by the quality (not quantity) of the student's participation in class.

There is a Canvas site for the course, but it will really only be used to record grades. Please do not pay any attention to any 'grade average' that Canvas might report. These are useless.

Problem sets are due *in class* on the day specified. I will not accept late problem sets. On the other hand, you will find that I am quite prepared to grant extensions, so long as they are requested in advance, that is, at least one day prior to the due-date. Extensions will not be granted after that time except in very unusual and unfortunate circumstancess.

Because I am so reasonable, exploitation of my reasonableness will be taken badly.

Time Expectations

- We expect to have about 36 meetings, so you will spend about 36 hours in class.
- Much of what we will be reading are original sources, and they are not easy going. So you should expect to spend about 2 hours per class reading the material for that class and reviewing material from the previous one. This accounts for 72 hours.
- You should expect to spend about 8 hours on each of the problem sets, on average, though some are longer and some are shorter. This accounts for 56 hours.
- You should expect to spend about 16 hours preparing for and completing the final (whether it ends up being in-class or take-home).

You should thus expect your total time commitment for this class to be about 180 hours.

In Class Behavior

Students may use laptops and the like to take notes in class or to access material we are discussing in class, but *all other use of computers, tablets, and mobile devices is prohibited during class*. This includes but is not limited to checking email, texting, and searching the web, even if the search is related to the course. I establish this rule not for my benefit, not even for yours, but rather for that of your peers.

In a study entitled "Laptop multitasking hinders classroom learning for both users and nearby peers", Faria Sana, Tina Weston, and Nichola Cepeda showed eactly that. It is not just that students who "multi-task" during class—check e-mail, text, or whatever—received significantly lower grades in the study than students who did not. This is not surprising, since the human brain simply cannot focus on very many things at one time. (If you're skeptical about this, then watch this video or perhaps some of these ones.)

Rather, the surprising conclusion was that students who were sitting *near* other students who were "mutl-tasking" *also* received significantly lower grades than students were who not. In fact, they were almost as distracted as the students who were actually doing the multi-tasking. There is thus evidence that "multi-tasking" does not only hurt the person doing it. It also harms the people around them. And that is the basis of my request that students not engage in such activities during class. If someone near you is doing so, you should feel free to ask them to stop.

Syllabus

The syllabus is approximate. We will proceed at whatever pace works for this particular group, and if enough of us get interested in some topic not on the particular route we are taking, we may digress for a bit. Please check the course web site for updates as we proceed.

25 January

- Introductory Meeting

27 January–10 February

- Richard Heck, "Formal Background for the Incompleteness and Undefinability Theorems" (PDF)
  This will be review for some, but for the rest it will get us a general sense for what Gödel's theorem says and how it is proved.
  If you're having a hard time understanding the diagonal lemma, then read this informal account, as well.

13 February–3 March

- Kurt Gödel, "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I", in *Collected Works* v. 1, ed. S. Feferman, J. Dawson, and S. Kleene (Oxford: Oxford University Press, 1986), pp. 144-95. (DjVu)
  Note that only the odd pages are in the DjVu; the even pages from this edition are in German.
  There are also inexpensive books available that have this paper in them, such as *The Undecidable*, edited by Martin Davis.

6 March–17 March

- Alfred Tarski, Andrzej Mostowski, and Raphael Robinson, *Undecidable Theories*, chapters I and II.
  You can skip I.6 and II.6, if you wish, or just skim them.

- Optional: James P. Jones and John C. Shepardson, "Variants of Robinson's Essentially Undecidable Theory R", *Archiv for mathematische Logik* 23 (1983), pp. 61–4 (Springer)

20 March–5 April

- George Boolos, *The Logic of Provability* (Cambridge: Cambridge University Press, 1993, Ch. 2 (DjVu).
- Optional: Martin Löb, "Solution of a Problem of Leon Henkin", *Journal of Symbolic Logic* 20 (1955), pp. 115–8 (JSTOR); Robert G. Jeroslow, "Redundancies in the Hilbert-Bernays Derivability Conditions for Gödel's Second Incompleteness Theorem", *Journal of Symbolic Logic* 38 (1973), pp. 359-67 (JSTOR).
  I encourage everyone to read the former, which is not too difficult. The latter is suggested only for those who are otherwise having an easy time with this material and are looking for a challenge.

7–28 April

- Solomon Feferman, "Arithmetization of Metamathematics in a General Setting", *Fundamenta Mathematicae* 49 (1960), pp. 35-92 (PDF).
  A poor quality PDF is available from the European Digital Mathematics Library.

Final Exam

- The final exam is scheduled for 17 May at 2pm.

Problem Sets

Problem sets will be due one week after we complete the relevant material. Dates below are thus estimates. But the problem sets will never be due *before* those. You are advised, however, not to wait until we finish the material to begin the problem set. You will be much happier if you start working on the problems as we cover the relevant material. Among other things, this will reveal to you, right away, if there is something you do not really understand.

1. First set of exercises for "Formal Background", covering sections 1–5: 15.1(i,ii), 15.2, 15.3, 15.5–15.8, and 15.10. Exercises 15.4 and 15.9 are optional.
   Due about 8 February.

2. Second set of exercises for "Formal Background", covering sections 6-13: 15.11–15.15, and 15.17–15.19. Exercise 15.16 is optional.
Due about 17 February.

3. First set of exercises for Gödel's "On Formally Undecidable Propositions": These are exercises 2.1-2.7.
Due about 1 March.

4. Second set of exercises for Gödel's "On Formally Undecidable Propositions": These are exercises 2.8-2.14. Exercise 2.11(ii) is optional (and a bit challenging).
Due about 10 March.

5. Exercisesfor Tarski, Mostowski, and Robinson, *Undecidable Theories*: These are exercises 6.1-6.4, 6.6-6.7, and 6.9.
Due about 24 March.

6. Exercises for Boolos, *The Logic of Provability*, Ch. 2: Specific exercises TBA.
Due about 12 April

7. Exercises on Feferman's "Arithmetization of Metamathematics in a General Setting": Specific exercises TBA.
Due about 5 May